Unit 7: Oscillations

In this unit, we investigate motion that results when an object is subject to a spring force. We have already seen the basic features of this motion: oscillations in position, velocity and acceleration as a function of time. We will examine this in greater detail, and also examine cases where the spring force is joined by a drag force (the damped oscillator) or with an external oscillating force (the driven oscillator).

Session 1: Simple Harmonic Motion

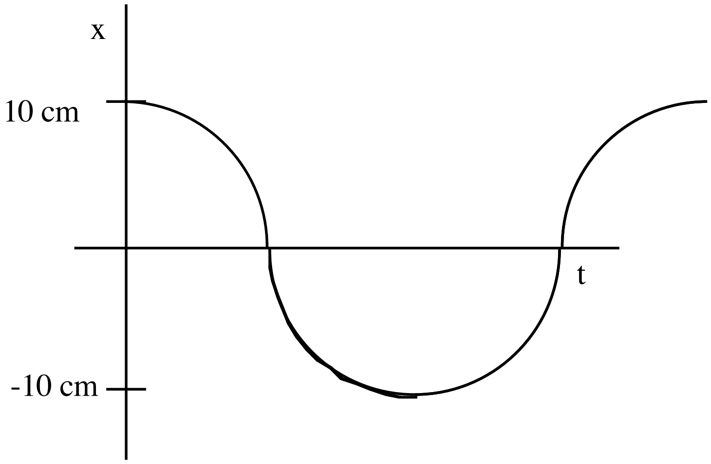
Motion that is governed strictly by a spring force is purely sinusoidal. We will develop some terminology that is used to describe this motion, and then see how theory relates this to the spring constant and the mass, the two fundamental parameters over which we have control.

First, let's look at the terminology, and use Mac Motion to make sure that we know how this relates to actual motion.

Guidebook Entry 7.1: Describing Oscillatory Motion

There are two quantities that are most commonly used to characterize oscillatory motion. These are the amplitude of the oscillation (how big it is) and the frequency of the oscillation (or equivalently, the period: how long it takes for a full oscillation).

The amplitude of an oscillation is defined as the maximum excursion from the equilibrium point (or the center point of the oscillation). What is the amplitude of the following oscillation?

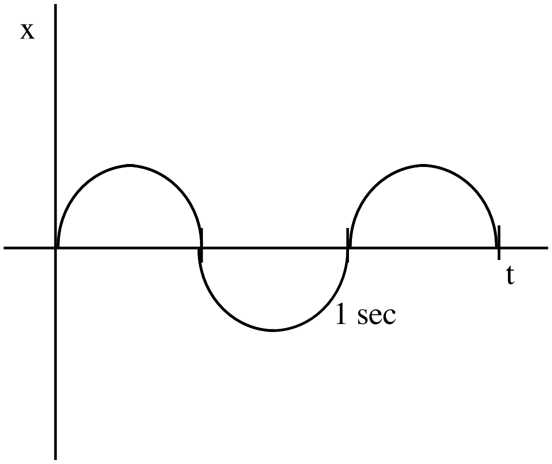


Using Mac Motion, create a graph from the motion of your own hand that as nearly as possible replicates the graph above (with the exception of an offset in the x value). Sketch your graph below.

Now move your hand in a way that gives an oscillation with twice the amplitude. Sketch the resulting graph below.

The period of oscillation is a measure of how much time elapses per cycle. The frequency is a measure of how many cycles occur per unit time. Given these definitions, how is the period (typically symbolized by T) related mathematically to the frequency (typically symbolized by f)?

What is the period of the following oscillation?



What is the frequency of that oscillation?

Move your hand in a way that duplicates this frequency of oscillation. Sketch the resulting Mac Motion graph below.

Move your hand in a way that gives twice the period of the graph I gave you. Sketch your Mac Motion graph below.

Move your hand in a way that gives twice the frequency of the graph I gave you. Sketch your result below.

The usual, general way of writing an oscillation is $x(t) = A\sin(\square t + \square)$. How does the amplitude relate to the parameters of this equation?

How does the frequency relate to the parameters of this equation?

What effect does the parameter ϕ have on the motion? Check with an instructor on the last three results.

The parameter ω is virtually identical to the ω used to describe circular motion and rotation in the last unit. In the case of oscillation, it is usually called the angular frequency of the oscillation. Although there are not obvious angles in the case of oscillation, in contrast to the rotation case, one often talks of the phase of the oscillation in radians, and ω gives the time rate of change of this phase in radians per second. It has the same relation to period and frequency of oscillation in both the rotational and the oscillatory cases.

In the next section, we will investigate the theoretical relationship between the angular frequency of oscillation ω and the other parameters of the oscillating sustem: m, k, and A.

Guidebook Entry 7.2: Theoretical Prediction of Angular Frequency

The fundamental principle that we will use in this analysis is Newton's second law: (F=ma), coupled with Hooke's law for the spring force (F=-kx). Combining these two gives us the differential equation of motion

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x.$$

Take the general expression for the position as a function of time

$$x(t) = A\sin(\omega t + \phi)$$

and differentiate it twice with respect to time. Then plug these expressions for x(t) and the second derivative into the differential equation.

What does your result tell you about ω?

Does this result tell you anything about A or ϕ?

Another way of interpreting the results of the previous exercise is to say that the angular frequency of oscillation of a mass on a spring is given by

$$\omega = \sqrt{\frac{k}{m}},$$

regardless of the amplitude or phase (starting time) of the oscillation. In the next exercise, you will verify this experimentally.

Guidebook Entry 7.3: Verifying the Frequency Formula

First, we want to predict the frequency of oscillation of a mass on a spring system. To do this, we need to know the spring constant k. To this end, calibrate your force probe. Then place a spring on the probe, and a 500 g weight. ZERO THE FORCE PROBE! Determine the force constant by moving the weight slowly up and down with the motion detector on the floor. Obtain k from the resulting graph of force versus position.

What frequency do you predict for the oscillation?

What period does this correspond to?

Start the mass oscillating, and measure the period, using a stop watch over several cycles, or using Mac Motion. Describe what you did, and give your results.

How does this compare with your prediction?

Change the amplitude of oscillation. Does the period change?

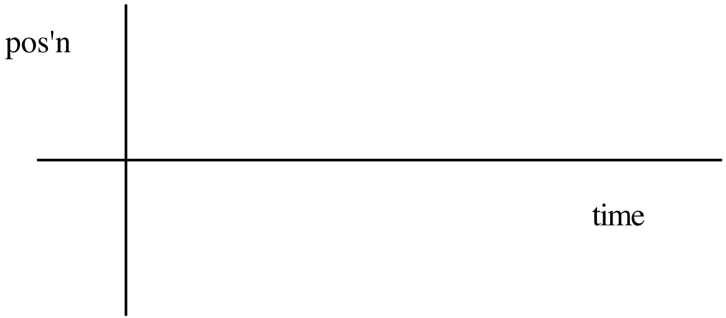
Now change the weight to one with a different mass. What should the new period be?

Measure this period. Does it agree with your prediction?

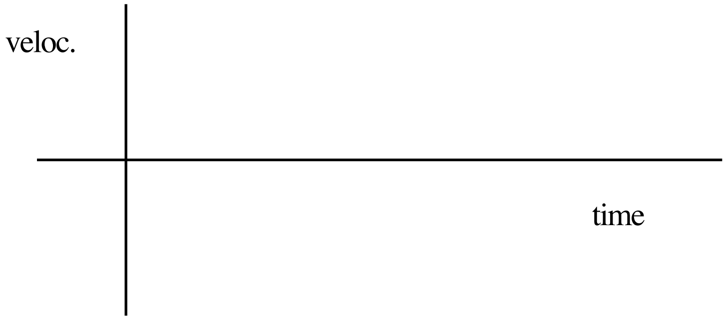
Session 2: Harmonic and Nearly Harmonic Motion—Details and Variants

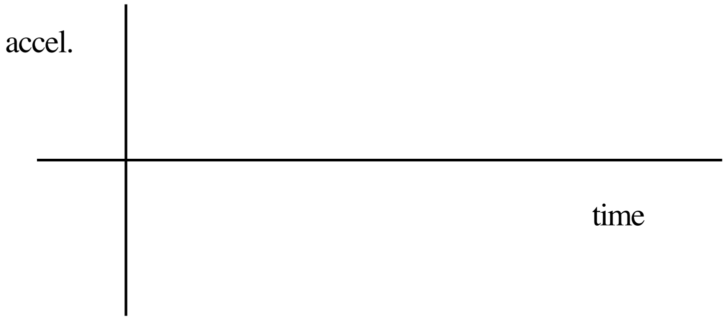
Let's start out this session with some of those aspects of oscillatory motion that reinforce the basic concepts of force, position, velocity, and acceleration.

> *Guidebook Entry 7.4: Position, Velocity, Acceleration, and Force—Predictions*
>
> We will be looking at an object subject to a spring force, $F = -kx$. What sort of motion results when an object is subjected to such a force? Answer in a few words and in a sketch of position versus time.

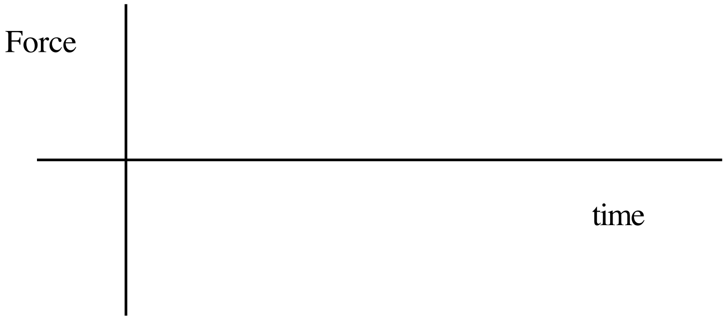


> What do the velocity and acceleration graphs look like? Mark the corresponding times, and explain to your partners how you related them.





What do you expect the force as a function of time graph to look like for the force exerted on this object? Explain. [There are at least two very different ways of getting this graph—can you think of more than one?]



In the next exercise, you will experimentally verify the relationships that hopefully you used to make your predictions above using the object hanging from a spring. But first, take care of the preliminaries of positioning the motion detector, calibrating the force probe, putting the spring and a 500 g. weight on the force probe, and zeroing the force probe after the weight is in place. Also, to improve accuracy (at the expense of a more jittery graph), set averaging for velocity and acceleration to 3 point.

Guidebook Entry 7.5: Position, Velocity, Acceleration, and Force—Observations

Start the weight moving, and use Mac Motion to record the position, velocity, acceleration, and force graphs. Print them out and attach them.

How did your predictions agree with the results for each of the following quantities? Be especially careful to look at phases of each graphs (i.e. starting conditions).

Position:

Velocity:

Acceleration:

Force:

Look at the Force versus time and Position versus time graphs simultaneously. How does force relate to position? What is this rule called?

Look at the Force versus time and acceleration versus time graphs simultaneously. How does force relate to acceleration? What is this law called?

In the next exercise, you will make quantitative observations of oscillatory motion, and see how they relate to the predictions that come from F = ma and calculus, especially the result that we found last session:

$$\omega = \sqrt{\frac{k}{m}}$$

Physicists call this type of oscillatory motion, where the frequency depends only on k and m, harmonic motion.

Guidebook Entry 7.6: Quantitative Analysis of Oscillatory Motion

In order to use the relationship above for the angular frequency of oscillation, we need to know the mass (which you can weigh, or trust the value stamped on the weight) and the spring constant. Find the spring constant. Describe first how you plan to do this, and then go ahead and measure it.

Given this value of the spring constant and a 500 g. weight, what should the resulting period of oscillation be?

Measure the period of oscillation directly. Does it agree with your prediction?

Given the value of the angular frequency that you determined above (either experimentally or theoretically, whichever you trust most), what should the *amplitude* of the *velocity* oscillation be? In other words, what is the maximum value the velocity should have? Discuss your result with an instructor.

Check your result with your own data. Does your maximum velocity agree with your prediction? If it is not close, check with an instructor.

In order to appreciate how simple the weight on a spring system is, we need to look at systems that are slightly more complicated. In the first of these, we will add a slight drag force to the spring force.

Guidebook Entry 7.7: Hooke + Drag = Damped Oscillations

First, we want to see the qualitative features of what are known as damped oscillations. A rubber band is *not* a very ideal spring. Put a 500 g. weight on a heavy rubber band hanging from your force probe, with the motion detector beneath. Use MacMotion to record the position as a function of time, and sketch below. Describe the graph in a few words—how does it differ from the oscillations we have observed thus far?

This motion includes sinusoidal terms (the oscillation comes from F = -kx) and a decreasing exponential amplitude. We know that decreasing exponentials result from drag forces, so let's deliberately add a small drag force to our spring. For a small change in the force law, we expect to see only small changes in the resulting motion. To enable us to look at these small changes, we need to look for a long time at the motion. Set the time axis in Mac Motion to run for 50 seconds. Take a 50 second set of data with the weight on the spring. Use the command in the data menu to transfer this data to "data B."

Now add a large square of cardboard to the bottom of the weight using the double stick tape (ask for some assistance with this). This will add a very small drag force, and change the system into what is usually called a damped oscillator. This damping is a form of friction—it "uses up" mechanical energy by turning it into heat. Run the same experiment. Sketch or attach your graph.

Does the period of oscillation change noticeably between the two experiments?

Use Excel to show that this function is qualitatively well represented by a function of the form $x(t) = Ae^{\square\square t}\sin(\square t)$. Adjust the values of □ and □ in your Excel sheet to get qualitative agreement with your rubber band data. Roughly how big is the constant □ compared to □ for the rubber band?

How about for your spring/cardboard data? (Notice that □ and □ have the same units, so such a comparison is valid

If you choose to continue in your study of physics, you will undoubtedly revisit this case, and study how to relate the various constants. It is not extremely difficult, and is covered as an optional topic in many introductory physics books if you are curious.

In the next case, we will change the spring force law into a law that is non-linear by attaching a string to the spring. In contrast to the damping example, the frequency of oscillation (or period) will no longer be a constant, but will depend on the amplitude of oscillation.

Guidebook Entry 7.8: A Non-linear Oscillator

Ask an instructor to add a string to your spring. This string has no effect when the spring is at equilibrium or compressed, but when the spring is stretched, it will prevent a portion a subset of the spring from stretching.

What do you expect the force versus distance graph will look like now? Explain your reasoning.

Use Mac Motion to make a graph of force versus distance, and attach or sketch below. If your graph doesn't look like your prediction, discuss this with an instructor.

Use Mac Motion to look at small oscillations, small enough that the string is always slack. Save this data into "data B." Does the position versus time look reasonably sinusoidal?

Now use Mac Motion to look at larger oscillations, large enough that the string is taut for a significant time each oscillation. Look at position, velocity, and acceleration versus time graphs. Do you see any significant deviations from sinusoidal behavior? Explain what you see.

Can you explain why the motion looks the way it does in terms of the force law?

Now compare the periods of oscillation of data A and data B. Are they the same? Can you make any sense of this? Discuss your thoughts with each other and an instructor.

This example illustrates the first effects of non-linear restoring (spring-like) forces. First we observe changes in period, and then we observe some small deviations from sinusoidal behavior—generally most noticeable in the higher derivatives, such as acceleration. If we add an external periodic force, such as shaking the top of the spring up and down in a regular way, we can get even more remarkable effects, effects that are at the heart of the study of non-linear equations most commonly known as chaos.