Unit 6: Circular Motion and Angular Momentum

In session 1 of the last unit, you investigated circular motion with bowling balls on the Harris Center floor. You found that circular motion resulted when you applied a constant force that was always perpendicular to the motion. In this unit, we will investigate this motion more carefully, and see how this relates to force. We will also look briefly at rotational motion more generally, and develop the rotational equivalents of force (which we call the torque) and momentum (which we call the angular momentum).

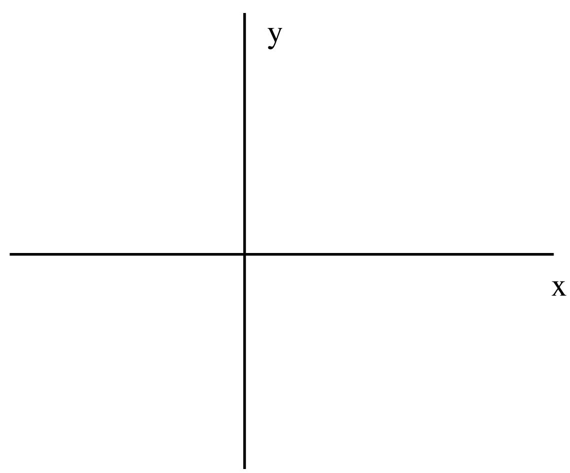
Session 1: Introduction to Circular Motion

In this session, we will develop some intuition about the description of circular motion in terms of the x and y components of the motion. We will start this by looking at the data you took in the Harris Center.

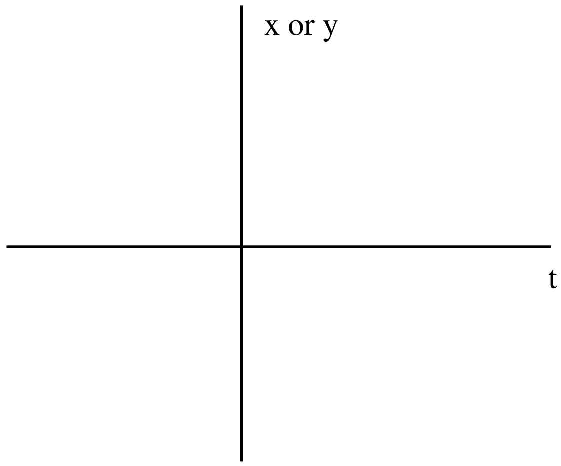
Guidebook Entry 6.1: Graphing Your Harris Data

First, take the circular motion data that you took in the Harris Center and enter them into a spreadsheet with the three columns labeled time, x component, and y component.

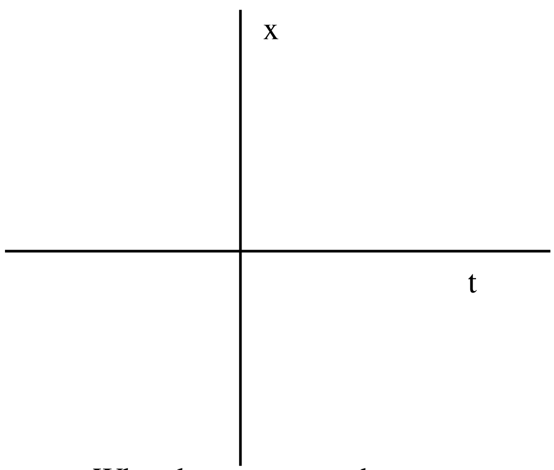
Now, graph these data as y vs. x, and sketch what you get below. Is this (within stretching of scales) what you saw on the floor?

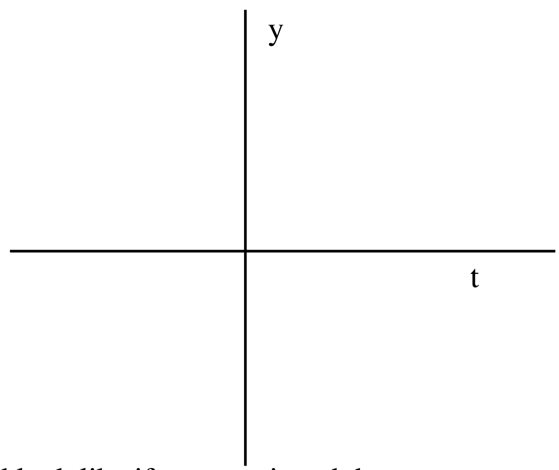


What do you expect either one of the components to look like if you graph them versus time? Sketch your qualitative prediction below:



Now take your actual Harris Center data and graph each of the components versus time. Sketch what you see below:





What do you expect these curves would look like if you continued the experiment and let the ball go around the circle several times?

Are the x and y curves identical? Describe any differences, and discuss your answer with an instructor.

Now that you have used your own rough data from the bowling ball experiment, it is time to make more precise measurements from a simple graphing exercise. We will simulate the motion by marking points on graph paper in a more ideal representation of circular motion than the bean bags on the floor of the Harris Center.

Guidebook Entry 6.2: Circles on Graph Paper

Take a protractor, and draw a circle the diameter of the protractor on graph paper, carefully centering the protractor at your origin. Mark points on this circle every twenty degrees, starting with the t=0 point along your x axis, and moving in the mathematical positive direction (counterclockwise). Record the x and y components of position for each of the points below:

Angle	x component	y component
0		
20		
40		
60		
80		
100		
120		
140		
160		
180		
200		
220		
240		
260		
280		
300		
320		
340		

Graph the y data versus x data using Excel, and sketch or attach below. Do you get the figure you expect?

Now graph the x data versus time, and the y data versus time. What functions do these look like?

You should have seen a clear similarity between the functions you graphed and the sine and cosine functions. In the next exercise, you explore this relationship a little more quantitatively.

Guidebook Entry 6.3: Mathematical Expressions for Circular Motion

Let's consider circular motion at a constant distance R from our coordinate system's origin. Write two mathematical expressions for the x and y components of position as a function of angle. Don't hesitate to ask for help with this.

x =

y =

Now rewrite these expressions in terms of a function of time, assuming the object is moving around the circle at a rate of one radian per second. (Recall that 2π radians = 360 degrees).

How would these expressions change if the object were moving around the circle at two radians per second? Or five radians per second?

How would you change these expressions if the radius of the circle were twice as big?

The usual expression for circular motion is something of the character

$$(x, y) = R(\cos \omega t, \sin \omega t)$$

where R is the radius of the circle, and ω is referred to as the angular velocity. It has units of radians per second, which is often considered just sec^{-1}, since radians are usually thought of as unitless. We will use this expression a great deal in the following sessions. In the next activity you will begin to investigate some of the connections between the angular velocity and the concepts of position, velocity and acceleration that we developed in our first unit.

Guidebook Entry 6.4: Angular Velocity and Speed--Relationships and Analogies

Imagine we are running around a circular track at a speed of 4 m/sec. The track is 200 m per trip around. How long does it take us to run once around the track?

There are 2π radians in one trip around the track. What then is our angular velocity?

What mathematical expression relates the speed and the circumference to the angular velocity?

Show that the relationship between speed and the circle's radius is even simpler.

Now imagine I have been steadily warming up, so my speed has increased to 6 m/sec some 300 seconds later. What is my new angular velocity?

If my speed (and therefore my angular velocity) increased steadily over those 300 seconds, what was the rate of change of my angular velocity? What do you expect is the name of this quantity?

It should be clear that the angular velocity is analogous to our ordinary linear velocity from Unit I. Complete the analogy then--what quantities are analogous to position and acceleration?

You should have discovered a natural analogy between position, velocity and acceleration and the corresponding quantities for rotational motion: angle, angular velocity, and angular acceleration. In this session, we have primarily considered what is known as *uniform* circular motion--that is, where the angular velocity is unchanging (angular acceleration is zero). The next session will focus exclusively on that case, and the third session will investigate primarily cases where the angular velocity is changing. This is, of course, more than just an analogy, since the distance along the circumference, the speed, and the rate of change of the speed is related to angle, angular velocity, and angular acceleration by a simple factor of the circle radius. Notice I was careful to say "distance along the circumference" and "speed" instead of position and velocity. This is because the speed may be constant for an object in circular motion, but the velocity definitely is not! Next session will start out wrestling with this puzzlement!

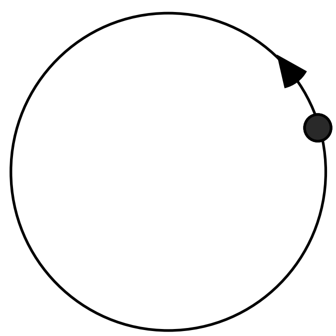
Session 2: Acceleration for Uniform Circular Motion

In this session, we will investigate how force relates to circular motion where the angular velocity is constant. In the process, we will discover some somewhat surprising relationships between position, velocity, and acceleration for circular motion that will help us understand the qualitative result you observed in the Harris Center for circular motion.

Guidebook Entry 6.5: Some Qualitative Observations Regarding Circular Motion.

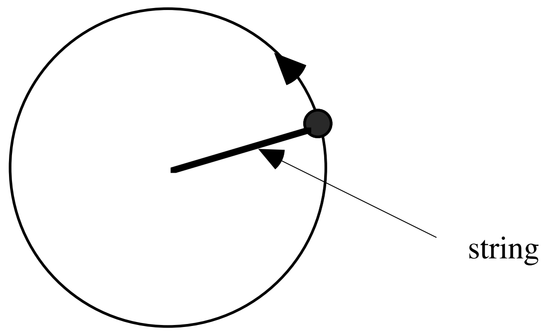
If an object is moving, and no forces are applied, describe the resulting motion:

For an object to move in any path other than a straight line, forces must be applied. For an object moving in a circle, what direction do you think the force must be applied? Sketch the direction of your prediction of the force on the diagram below, and explain.



We know that we can spin a ball attached to a long string. This gives us some hints as to the direction of the force. But first, we have to know a bit about how strings work. Hold one end of a string, and have a partner hold the other end. Can you exert a force on your partner through the string in a direction perpendicular to the string? Explain.

If the force exerted by a string can only be in the direction of the string, what must the force direction be for an object in circular motion? Use the diagram below as a reference if you wish. If you wish, you can ask for an object on a string to swing around.



How would you characterize the relationship between the velocity and the force for circular motion? Does this make sense in terms of the way you produced circular motion at the Harris Center?

In the last session, we discovered that the expression for position as a function of time for an object in circular motion looked like

$$(x,y) = R(\cos \omega t, \sin \omega t)$$

where R is the radius of the circle, and ω is the rate at which it goes around the circle in terms of radians per second, the angular velocity.

But motion is related to force through the acceleration, *not* through the position. So, for us to be able to extend our F=ma mechanics to circular motion, we have to see how the acceleration relates to the parameters of circular motion. In the first exercise, you will use the formula above in a spreadsheet to produce artificial circular motion data, and then numerically calculate the velocity and acceleration.

Guidebook Entry 6.6: Numerical Calculation of Velocity and Acceleration

Create a spreadsheet with the following columns: time, x, y, x velocity, y velocity, magnitude of velocity, x accel, y accel, and magnitude of acceleration.

First, fill in a column of 100 time values running from 0 to 10 seconds. Then calculate the x and y values for circular motion using the formula above, with a radius of 5 and an angular velocity of 2 radians per second. Recall that Excel assumes the argument of the trigonometric functions to be in radians.

Next, enter and fill down the formulae for the x and y components of velocity. In particular, for the x velocity, if this is in cell D3, and the x column is B, this would have the formula

= (B3-B2)/.1

assuming that .1 is the time interval between cells.

Notice that the top most row of these velocity values will be empty or contain nonsense, since it will refer to non-existent cells.

Now enter the formula for the magnitude of the velocity, that is, the speed. Since you need to use the Pythagorean theorem, you will need the square root function, which is SQRT( ) where the appropriate arguments, be they cell references or numbers or algebraic combinations thereof, go in the parentheses. Fill this down.

What do you notice about the speed as a function of time?

What do you expect happens to the speed if you cut R in half but keep the same angular velocity ω? Check this using your spreadsheet.

Return R to a value of 5. What do you expect happens to the speed if you now cut ω in half? Check this using your spreadsheet.

Considering your results from above, can you guess an equation that relates the speed to □ and R? Do the units of your formular make sense? Does it agree with your results from the last session? Check your answer with an instructor.

What is the formula for the x component of acceleration in terms of the values of the velocity? How would you write this in a way appropriate for your spreadsheet?

Insert the formulae for the x and y components of acceleration, as well as the magnitude of the acceleration. Fill these formulae down. What do you notice about the magnitude of the acceleration?

Use the same sort of tests you did before (cutting □ and R in half) to see how the magnitude of the acceleration depends on R and □. Can you guess an equation that relates the magnitude of the acceleration to R and □? Once again, check units and check your result with an instructor.

This acceleration that is associated with circular motion must arise from a force, if we are to continue to believe Newton's Second Law F = ma. What force is necessary to keep an object moving along a circular path, in terms of the mass m, the angular velocity ω, and the radius of the path R?

Don't lose this spreadsheet, because you will use it once again. But before we do, it is time to verify experimentally the final result in the last exercise, that is, that

$$F_{centripetal} = mR\omega^2 = m\frac{v^2}{R}.$$

Consideration of this force that causes circular motion is so common that it is given its own name, the centripetal force.

Guidebook Entry 6.7: Experimental Verification of Centripetal Force

At your station, you will find a rotational demonstration apparatus. This device allows you to observe the black weight hanging by a loop of string as it undergoes circular motion. This weight has the bottom shaped into a point, which we will position over the metal pointer fixed to the base as a way of indicating the radius of motion.

First, simply spin the rod, and observe what happens to the black object as you spin it at a rate of a couple of revolutions each second. Describe in particular what happens to the spring:

This device is constructed so that the spring is the only possible source of centripetal force. This is particularly convenient, since the spring force depends only on how far the spring is stretched, which we can later measure with a force scale.

With the spring disconnected, adjust the height of the black weight and the position of the metal pointer such that the black weight just clears the pointer, and is immediately above it when it is hanging freely. In this position, the string exerts only a vertical force, and therefore can provide no component of centripetal force.

Now reattach the spring, and spin the device faster and faster until the black weight is exactly over the pointer. Measure how long it takes for the weight to rotate a fairly large number of turns (say ten or twenty). You will need to keep twisting the center rod a bit to maintain a reasonably constant rotation speed. Record your observations below.

Time:

Number of Revolutions:

What then is the angular velocity ω in radians per second?

What is the radius of the circular motion?

What is the mass of the black weight?

Using the equation for centripetal force from the previous page, what should the centripetal force have been?

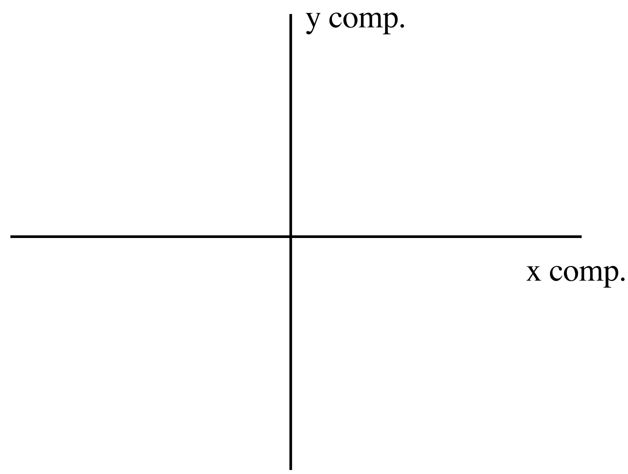
Using a spring force gauge, measure the force necessary to stretch the spring out so that the black weight is just over the pointer. Is it what you predicted from the centripetal acceleration formula?

Finally, we would like to see that our calculated *directions* for velocity and acceleration agree with what we guessed, and what we observed. In particular, we observed that the centripetal force was directed in toward the center of circular motion. We hope that this is also the direction of acceleration that we calculated.

Guidebook Entry 6.8: Directions of Position, Velocity and Acceleration

Pick some arbitrary time in your spreadsheet, preferably when the position is not nearly aligned with one of the axes. What are the values of position (x,y), velocity (v_x,v_y), and acceleration (a_x,a_y)?

Graph each one of these values on the same set of axes:



How would you describe the relationship between the direction of the position and the direction of the velocity? Does this make qualitative sense (think in particular of the constant radius required for circular motion)?

How would you describe the relationship between the direction of the velocity and the direction of the acceleration, and therefore force? Does this make sense (consider especially what you saw in the Harris Center)?

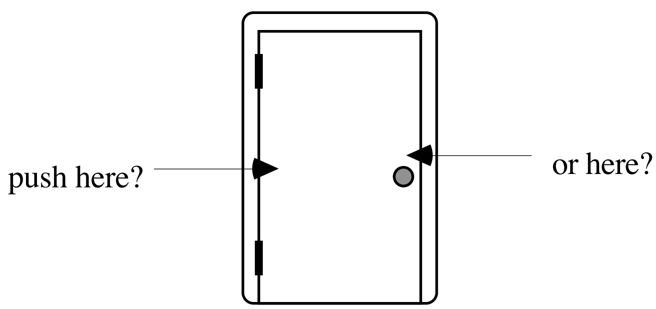
Finally, how would you describe the relationship between the direction of the position and the direction of the acceleration, and therefore force? Does this make sense (consider especially the spring force law)?

Session 3: Torque and Angular Momentum

In this session, we discuss some of the general features of rotation, especially when the angular velocity is *not* constant. We will develop some concepts through consideration of extended, solid objects that we can rotate, in contrast to the abstraction of point-like objects that we have implicitly discussed up until now. In the process, we extend the concepts of force and [linear] momentum to their rotational analogs, torque and angular momentum. And while we develop these concepts considering extended, solid objects, they also will be applicable to the motion of point like objects, such as an object in circular motion.

Guidebook Entry 6.9: Torque—Let's Do the Twist Again!

Imagine that you wish to open a heavy door with the least possible effort. Where should you push? Explain.

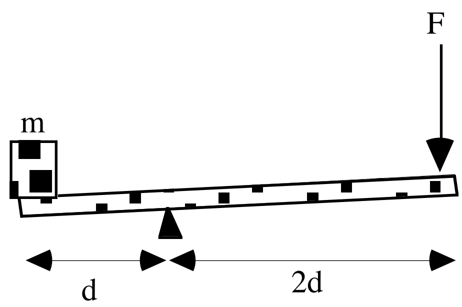


(Ok, ok. If you look closely at the hinges you might object that the door won't open if you push it. Imagine that it is free to move inward).

How does your ability to open the door depend on the amount of force you apply, at least qualitatively? In what direction do you need to apply that force (i.e. direction of the force relative to the door)?

Make a simple mathematical combination of force and distance from the hinges that would be a useful predictor of the relative effectiveness of a door-opening push.

Consider now the situation in which you are using a lever to lift a heavy object, as shown:



We are pushing steadily down with a constant force on the right side. This force is just large enough to make the lever rotate around the triangular fulcrum. If we push the right side down 1 cm, how far does the left side come up?

Are there any obvious sources of friction in this system? Do you expect mechanical (i.e. not including heat and sound) energy to be conserved? Explain.

What is the work done by the force on the right of the lever if the right side is pushed down 1 cm? (At this point we don't know what F_{right} is, so just leave it as a variable).

What is the work done by the lever on the weight on the *left* if the *right* side is pushed down 1 cm? Notice that we do know F_{left} = -mg.

If mechanical energy is conserved, then the work done by the force on the right is equal to the work done on the weight. If the mechanical energy is conserved, what must the force on the right be?

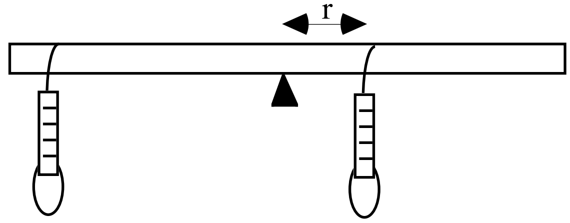
How does this force compare with what you would need to apply to lift the heavy object straight up (i.e. without the lever)? Can you understand the origin of the phrase "mechanical advantage?"

This is the principle behind the simple lever. We can raise a heavy object by using a small force (in this case, the force F required to raise the object is only half of its weight. Note however that to raise the object 1/2 cm we had to apply half the force for twice the distance, 1 cm). In this case, we aren't actually moving vertically, but taking a mostly vertical portion of rotational motion, and the applied force plus the fulcrum tends to twist the object, rather than simply accelerate it.

Now that we have a prediction for how this "twisting force" (which is called torque and symbolized by τ) should depend on force and the distance away from the center of application, let's test this in a real situation. Rather than seeing how effectively we can rotate something, we will instead apply two different torques such that they cancel one another out. In this way, you can evaluate experiementally the relative importance of force and distance.

Guidebook Entry 6.10: Balancing Torques

We have two levers that consist of metersticks on a fulcrum which will allow you to quantitatively check the dependence of torque on distance from



the fulcrum and force. One way to do this is to apply a constant force (say 5 N) with a force gauge far to one end (left end in my picture). If you prefer, you may hang a 500 g weight there. Then see what force is necessary to balance that force on the left by applying a force on the right at a variety of different distances r from the fulcrum.

Distance (r)	Force on right

Since the 500N at 0.5 m on the left is applying a constant torque, the combinations of force and distance on the right must all be supplying an equal but opposite torque to achieve balance. Can this be explained quantitatively by your formula for torque? If not, try to correct your formula. Discuss your result with an instructor.

Just as we found that forces caused accelerations (that is, changes in velocity), we expect torques will have some similar effect on rotational motion. We won't try to prove the rotational equations for all cases, but rather motivate them by looking at a point particle. We will simply rewrite Newton's 2nd Law for rotational, rather than linear motion. In the process, we'll find a way to generalize mass for rotational motion.

Guidebook Entry 6.11: A Twisted Version of Newton's Second Law

Start by rewriting the expression of Newton's second law as force being equal to mass times time derivative of the velocity:

Now multiply each side by r, the distance from the center of rotation. Here we're implicitly assuming that this particle is constrained to rotate somehow; try not to let that bother you. On the left hand side of your equation you should recognize the torque τ= rF, which you found from the experiments above.

On the right side of your equation, we have velocity, which is a concept appropriate for linear motion. How is the speed (magnitude of velocity) for a particle at a distance r from the center of rotation related to the angular velocity ω?

Solve this expression for v, if necessary, and then substitute that into your modified Newton's second law equation on the previous page. Now make the assumption that the radius r really doesn't change for this object (i.e. that it really is rotating, and not in linear motion), and pull the r out of the derivative expression.

Rewrite your final expression using the concept of angular acceleration, which we saw briefly in the first session of this unit:

$$\alpha = \frac{d\omega}{dt} .$$

You should now have an expression that contains only angular variables. If we say the torque plays the role of force, and the angular acceleration plays the role of the linear acceleration, what is the angular equivalent of mass? Check your result with an instructor. This quantity is usually called the moment of inertia, or sometimes the rotational inertia (which I prefer, since it clearly mentions rotation)

It should not be surprising, that if one applies a constant torque to an object, it will change its rotational speed, its angular velocity, in much the same way as a force causes a linear velocity to change. We can then simply transform some of our *F=ma* solutions for constant force into constant torque solutions for the equation $\tau = I\alpha$:

$$\theta = \frac{1}{2}\alpha t^2 + \omega_0 t + \theta_0 \text{ and } \omega = \alpha t + \omega_o .$$

We will use these in our next exercise to experimentally verify our expression for the rotational inertia.

Guidebook Entry 6.12: Testing the Rotational Inertia Concept

The first test of rotational inertia is rather subjective. You will take two one-meter sticks, hold them together (or tape them together) and try to rotate them back and forth in your hand about the midpoint of the stick, like a baton twirler. Now try this with a two-meter stick (same mass, twice the radius). Which stick is harder to rotate?

Predict using the rotational inertia concept which object will be harder to spin. How much harder (as a ratio)? Does this seem at least plausible?

Now let's try to be quantitative about the rotational inertia. The rotational apparatus you used before has been modified to allow you to look at rotational inertia. The rotational inertia of the apparatus is dominated by the two metal blocks on the cross rod. The total rotational inertia is just the sum of all the individual rotational inertias, i.e.,

$$I = \sum_i m_i r_i^2 ,$$

and hence is approximately just the sum of the two terms due to the metal blocks. How large is this rotational inertia?

You will be accelerating this by applying a 2 N force to the string. This string is at a radius of one half the diameter of the support rod from the center of rotation. What constant torque does this correspond to?

You will be applying this torque for a period of 5 seconds. How fast should the apparatus be rotating after this time in radians per second, assuming it has started at rest?

Now do the experiment. Apply the force with a force gauge for 5 seconds, release the string, and then measure the time required for the next few revolutions. Use this time to calculate the angular velocity ω. Does this agree reasonably with your prediction?

Now move the metal blocks in to half the distance from the end. How much faster should the apparatus end up rotating?

Do the experiment now with this smaller rotational inertia. Record your results. Does this agree with your prediction?

For regular linear motion, we found we could solve problems either by using F = ma directly, or by finding conserved quantities such as energy and momentum. We can easily develop such a conserved quantity for angular motion as well. To gain a little direction, let's recall what we did for [linear] momentum. We rewrote F=ma in terms of the derivative of a single quantity:

$$\begin{aligned} F &= ma \\ &= m\frac{dv}{dt} \\ &= \frac{d(mv)}{dt} \\ &= \frac{dp}{dt} \end{aligned}$$

This implies, given Newton's third law (equal and opposite forces between two objects) and no external forces, that the momentum is a conserved quantity. We'll follow this pattern for a new quantity relating to angular motion.

Guidebook Entry 6.13: Angular Momentum

We will simply start with the angular version of F = ma that we derived earlier, $\tau = I\alpha$. Rewrite this so the derivative is explicit, and then absorb the moment of inertia term into the derivative, just as was done above for the linear momentum case.

What quantity is conserved for angular motion in the absence of external torques? This quantity is called the angular momentum, and is usually symbolized by *L*.

We will at times want to calculate the angular momentum of a single, orbiting particle (i.e. one in circular motion). Rewrite your previous expression in terms of r, m, and v.

$L =$

We also might not have the linear velocity, so rewrite the previous expression in terms of r, m, and the angular velocity ω. Check your results with an instructor.

$L =$

One last equivalent expression. How can you write the angular momentum in terms of the linear momentum *p* and the distance from the center of circular motion *r*?

$L =$

To summarize and display the analogies we've created:

	linear	**rotational**
	position, x	angle, θ
	velocity, v	angular velocity, ω
	acceleration, a	angular acceleration, α
	force, F	torque, τ
	momentum, p	angular momentum, L
	mass, m	rotational inertia, I
		$I = \sum_i m_i r_i^2$
Newton's 2nd law	F = ma	$\tau = I\alpha$
Conservation	$\frac{dp}{dt} = 0$	$\frac{dL}{dt} = 0$
Constant acceleration	$x = \frac{1}{2}at^2 + v_0 t + x_0$	$\theta = \frac{1}{2}\alpha t^2 + \omega_0 t + \theta_0$
	$v = at + v_0$	$\omega = \alpha t + \omega_0$