Unit 5: Motion in Multiple Dimensions

Thus far in the course, we have restricted our attention to situations in which an object moves only in one dimension. For example, we considered falling objects that moved only vertically, or objects rolling in a straight line, or a mass moving only up and down on a spring. While this is very useful for a number of purposes, most motion in the real world does not have this constraint, but rather is free to move in two or three dimensions. Fortunately, we are able to retain virtually all of the structure that we have developed in one dimensional systems; we will continue to be able to solve problems using either straight F=ma techniques (solving the differential equation of motion) or using conservation principles (such as noting that if there are no external forces, $p_{before}=p_{after}$).

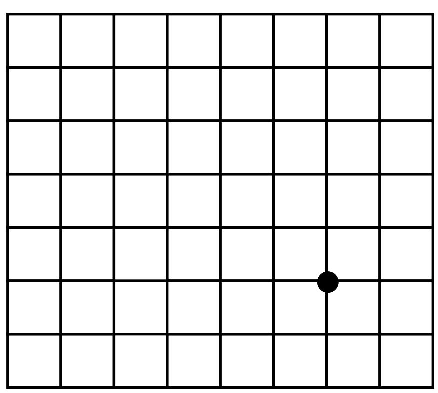
In the first session, you will get a chance to experience this in a very direct way by rolling bowling balls on a smooth surface, and letting them execute motion in two dimensions.

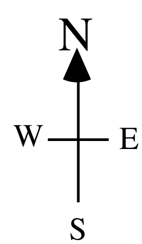
Session 1: Introduction to Two Dimensional Motion

To describe motion in multiple dimensions, motion that is not constrained to lie in a straight line, we need to describe position on a two-dimensional surface, or in three-dimensional space. In this first exercise, you wrestle a bit with the concepts necessary to do this.

Guidebook Entry 5.1: Getting From Here to There

Below is a schematic map of the city of Grinnell. Imagine that we are starting at the corner of 8th and Park, where the black dot is. I want you to get to my house, which is near the corner of 12th and West. I give you the following directions: go four blocks north, and then three blocks west. Find the location of my house on the map below, and mark it with an "x."





Would you have gotten to a different location if you had first moved three blocks west and then gone four blocks north? In other words, does the order matter?

Draw in a single arrow from the original black dot to my house. This arrow also contains the information of "three blocks west, four blocks north."

We refer to a displacement that contains both a distance and a direction, or equivalently, two components (i.e. a north-south component and an east west component) as a vector. For most of our practical purposes, we will refer to vectors in terms of their components. The convention that is used most commonly is that the two components are given sequentially in parentheses, first the right-left (E-W) component, and then the up-down (N–S) component. Displacements up or right are by convention positive, and down or left negative. Given these conventions, what is the displacement from the black dot to my house in units of blocks?

displacement = (,   )

Use the Pythagorean theorem to determine the distance "as the crow flies" from the dot to my house.

What is the angle of the displacement relative to directly north?

In this last exercise, you learned the two basic ways of describing a vector: by components, or by length and angle (usually called magnitude and direction).

In the next exercise, you will experimentally measure some displacements, and the time derivative of these displacements (which is the velocity, also a vector).

Guidebook Entry 5.2: Constant Velocity in Two Dimensions

In this exercise you will extend our one-dimensional experiment of rolling a candlepin ball to two dimensions. Before you do that, how would you describe the characteristics of free rolling in one dimension?

Sketch a graph of position versus time for one dimensional motion of a freely rolling ball.

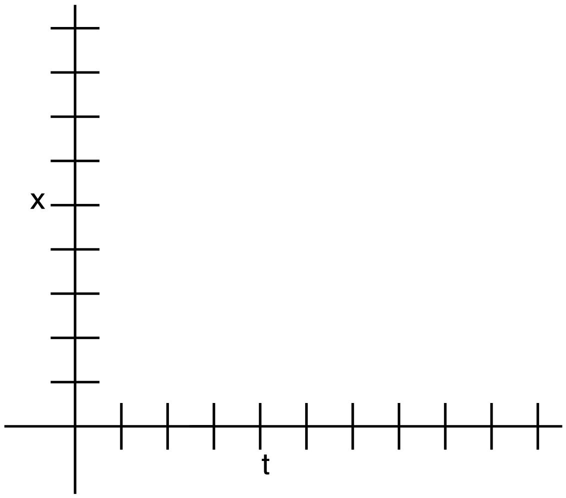
How does the velocity of the ball relate to the graph?

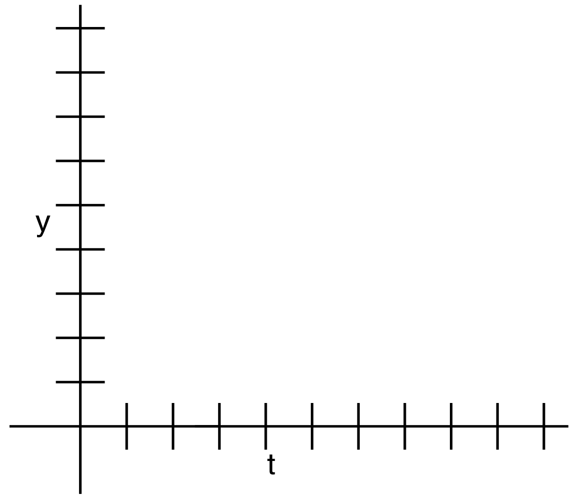
You will use the floor boards to define a coordinate system, i.e. the directions of the two components. Roll a ball at an angle of roughly 30˚ to the boards. Mark the position of the ball at one second intervals using the bean bags. Measure the position of the bags using a two-meter stick in terms of x (along the boards) and y (perpendicular to the boards) components, as well as the distance along the motion.

time	x component	y component	distance along motion

Does the distance along the motion agree with what you get from Pythagoras?

Graph the position versus time for each component below:





What is the velocity of each component?

Use Pythagoras to determine the velocity in the direction of motion.

What is the velocity from your "distance along motion" data?

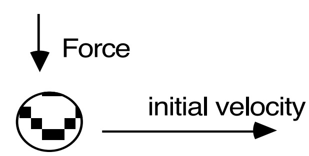
Do your two velocity values (for velocity along the direction of motion) agree?

What you should have observed in the last exercise was that once we move to two dimensions, Newton's first law (in the absence of force, velocity is constant) is also true by components. In other words, if there is no force acting on an object, it will have a constant speed in the x direction and a constant speed in the y direction. We can manipulate the components of the velocity to produce a magnitude, just as we did with position.

This naturally brings up the question of what happens when we introduce a force. The answer to this question depends not only on how large the force is, but also on the direction of the force. In the next exercise, we will apply a constant force with a constant direction.

Guidebook Entry 5.3: Constant Force Perpendicular to Initial Motion

In this exercise, you will roll a bowling ball along the direction of the boards (x direction) and apply a force (tapping with a stick) in the y direction, as shown.



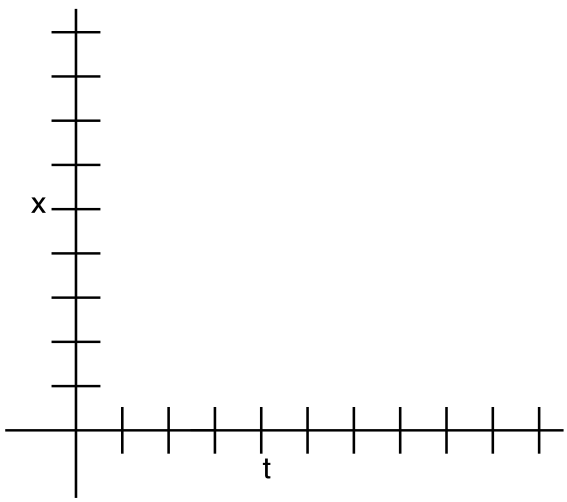
What do you expect the path of the ball to look like? Sketch your prediction below:

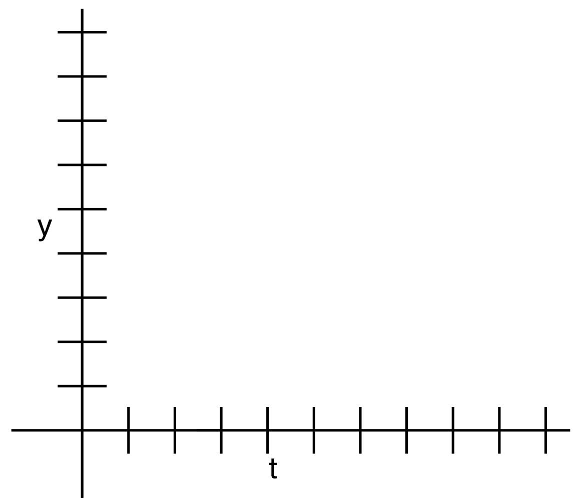
Now, try this experimentally. Roll the ball along the board direction, and tap with the stick perpendicular to the board direction. Regardless of where the ball goes, keep tapping perpendicular to the board direction. As before, mark the position of the ball every second with bean bags. Does the path agree with your prediction? Sketch it below.

Measure the positions of the bags (relative to a fixed origin) and record below in x,y components.

Time	x component	y component

Graph the x and y components of position below:





From your experience with one dimensional motion, which of these graphs is most nearly consistent with zero force, constant force, a spring force, or a drag force?

What you should have discovered above is that the motion is related to the force by components. In other words, if there is no force in the y direction, then the acceleration is zero in the y direction. If there is a constant force in the x direction, then the acceleration is constant in the x direction. We will later deal with this more quantitatively, using the multidimensional version of Newton's second law:

$$F_x = ma_x$$
$$F_y = ma_y$$

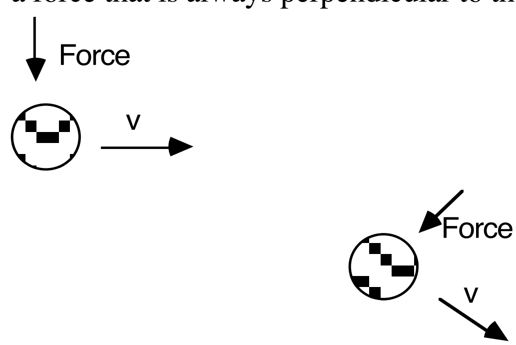
where the x and y subscripts indicate the x and y components of that particular variable. These component equations can be written in a shorthand way:

$$\vec{F} = m\vec{a} \, .$$

The theoretical solution of a multi-dimensional problem is no more difficult than the corresponding one-dimensional problems. For example, if the force is constant, then we have just position functions that are quadratic in time. However, if the components of the force are not constant, then the solutions are in turn more difficult. For example, even the subtleties of change in direction can be misleading. For example, if one has a constant force, but makes that force change direction, the motion is qualitatively different.

Guidebook Entry 5.4: Constant Magnitude Force Perpendicular to Velocity

If now you apply a force that is always perpendicular to the velocity as



shown, what do you expect the resulting motion to look like? Sketch your predicted path below:

Roll the ball initially along the board direction, and tap it with a constant intensity always perpendicular to its motion. Drop bags every second. Does the path agree with your prediction? Sketch it below.

Record the positions of the bean bags relative to a fixed origin. You will analyze these data in a future class, but you should check with an instructor to make sure you have usable data (especially to make sure you have signs correct).

Time	x component	y component

Your experiment should suggest that a force that is constant in magnitude but perpendicular to the velocity results in circular motion. This is a very important result that we will use in a variety of contexts.

Session 2: Separation into Components

In this session, we will investigate the situations in which Newton's second law, F=ma, can be easily separated into two (or three) components. You saw a hint of this in the exercise in the Harris Center, where a force applied in one direction seemed only to affect the motion in that direction. This separation happens in many (although not all!) cases, and allows us to solve a two (or three) dimensional problem as if it were simply several one dimensional problems. The only connection between the dimensions is the time, the common element that allows us to combine the two into a single motion.

In the Harris Center, you looked first at zero force, one dimensional motion at an angle to your coordinate system, and saw how you could break that up into components. Our first exercise today does a similar thing with constant force, one dimensional motion at an angle to your coordinate system.

Guidebook Entry 5.5: Linear Motion on an Angle

First, as an exercise in recalling constant force motion, predict the position of an object of mass 2 kg that is starting at rest at x=0, and subject to a 1 N force. You may wish to recall

$$F = ma$$

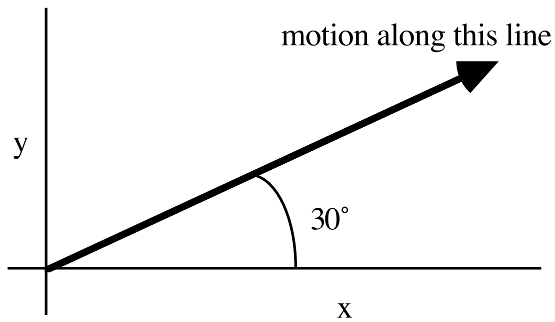
and

$$x(t) = \frac{1}{2}at^2 + v_o t + x_o$$

for motion under a constant force.

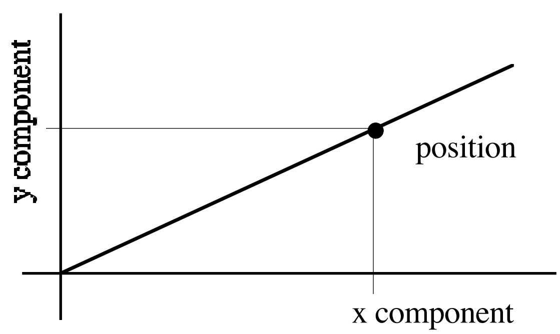
time (sec)	position (m)
0	
1	
2	
3	
4	
5	
6	
7	

Now imagine that this motion is on a two dimensional surface, much like the floor of the Harris Center. Imagine that the force is at an angle 30˚ off of the x axis, as shown



Take a sheet of the "10 mm to a cm" graph paper. Draw a line at 30˚ to the x axis; this will be the direction of motion. Using a ruler, mark points along this line that correspond to the motion you calculated above, using a convenient scale of your choosing.

Now take the x and y components of the motion by reading off of your graph, as shown below:



Fill the values into the table below:

time (sec)	x (m)	y (m)
0		
1		
2		
3		
4		
5		
6		
7		

Use Excel to graph both the x and y components versus time. Do they look like constant acceleration graphs?

How do you guess the x and y accelerations to be related to the acceleration along the direction of motion?

Calculate the acceleration associated with the x component and the y component from your data.

x acceleration:

y acceleration:

total acceleration:

Does it agree with your prediction?

Are the x, y and total acceleration in agreement with the Pythagorean theorem?

Summarize the relationship between the total acceleration and the components of acceleration.

What you have seen is that the acceleration in the component directions is equal to the appropriate sine or cosine function times the total acceleration, just as the position vector is related to its x and y components. Because force is related to acceleration through F=ma, we can say the same for the components of the force. This is a central theme to problems in multiple dimensions. We find the components of the force in the various dimensions, and solve them individually.

In the next exercise, we will observe a situation in which the forces in the x and y dimensions are very different. In particular, we will examine two dimensional motion under the influence of gravity, where the vertical component of the force is -mg, and the horizontal component of the force is zero. This type of motion is referred to as projectile motion.

Guidebook Entry 5.6: Projectile Motion

We will look at the motion of an object under freefall, but without confining the motion to a straight vertical line. If there is no force in the horizontal (x) motion, what equation describes x as a function of time?

x(t) =

If the force in the y dimension is -mg, what is the acceleration in the y direction?

What is the equation that describes y as a function of time?

y(t) =

Combine the two equations, eliminating the time variable. Rearrange the result in terms of y as a function of x:

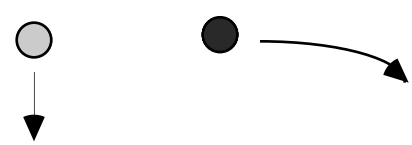
What is the shape of the path that the object follows?

Throw a bean bag through the air. Does the path qualitatively look like your prediction?

You should have predicted that the motion in the x direction is constant velocity motion, and in the y direction is constant acceleration. We have a device that demonstrates this, and shows that the motion in the y direction is unaffected by motion in the x direction.

Guidebook Entry 5.7: Quantitative Analysis of Projectile Motion

The device will simultaneously release an object to fall directly downward, and will shoot a second object exactly horizontally, like so:



Which object do you expect will hit the ground first? Why?

Ask an instructor for assistance with the demonstration apparatus. Watch and listen carefully to see if you can tell which object hits the ground first. What do you observe?

We have made two computer videos of the apparatus in action, which are on the hard disk of your computer. First run each one of the videos frame by frame. One of them has a larger initial horizontal velocity than the other. What can you say qualitatively about the vertical motion of the two balls?

Choose one of the videos to analyze in detail. Then run the video frame by frame. Mark the position of each of the balls on the screen with a washable marker. Also mark the horizontal and vertical references provided by the cinder blocks in the background.

Now measure (with a ruler) the x components of the positions of the right ball. Record those positions in the following table:

Frame number	x position

What do you expect a graph of x position as a function of time should look like? Explain.

Use Excel to graph the x position as a function of time. Sketch the graph below. Does it agree with your prediction?

Now measure the y position of the right ball, and record as a function of frame number.

Frame number	y position

Do these data look like constant acceleration in the y direction? How do you know?

We have seen that we can rewrite $\vec{F} = m\vec{a}$ as $F_x = ma_x$, $F_y = ma_y$ and $F_z = ma_z$. As in one dimension, we couple them with some force law, and solve for motion, now by components. We will concentrate on cases where the force law also separates into components, so we simply have three one-dimensional problems to solve, which are linked only by time. However, you should know that it is *not* required that F_x depends only on x, v_x, and perhaps t--it *may* depend, say, on v_y, which makes solution of the resulting equations *very* messy; we will avoid these cases!

Session 3: Energy and Momentum—Vectors and Scalars

In this session, we will address the question of how to generalize momentum and energy into multidimensional concepts. As a rule of thumb, we will find the same criteria determine applicability of energy and momentum conservation in multiple dimensions as in one dimension. That is, if there are no external forces, then momentum is conserved. If the forces depend only on position or are constant, then energy is conserved.*

First, let's reconsider some simple one-dimensional cases to get some clues as to the nature of momentum and energy in multiple dimensions.

Guidebook Entry 5.8: Momentum in One Dimension

Imagine that two equal mass carts are running toward one another at the same speed. They have Velcro pads facing one another, and stick when they collide. Do you expect momentum to be conserved in this collision (i.e. are there any external forces)?

Calculate the total momentum before and after from the velocities and the mass. Is momentum conserved?

Does the direction of motion matter when you are calculating momentum? Explain, and discuss your results with an instructor.

* To be completely accurate, for the energy case, one has to apply a vector calculus test to the force function to be sure energy is conserved. This tests to see if the work done in getting from point A to point B is independent of the route traveled. In practical terms examples that fail to conserve energy just because of this requirement are rare in the real world, and we will not deal with any force laws that are just functions of position but fail to satisfy this more stringent test of energy conservation.

Because momentum is direction dependent, it must translate into a vector quantity when we generalize this to two and three dimensions. In other words, like position, velocity, and acceleration, momentum has x, y, and z components. When we say that momentum is conserved in multiple dimensions, this means that each component is conserved individually.

Guidebook Entry 5.9: Energy in One Dimension

Imagine again that two equal mass carts are running toward one another at the same speed. Once again, they have Velcro pads facing one another, and stick when they collide.

Does the Velcro force depend on more than just position? In particular, is the force different when you push the two carts together than when you try to pull them apart (feel free to try this if you are unsure, and consider especially force directions)?

Do you expect total energy to be conserved given the nature of the Velcro force?

What is the kinetic energy (in terms of the mass m and velocity v) of the left cart?

What is the kinetic energy (in terms of the mass m and velocity -v) of the right cart?

What is the kinetic energy (in terms of the total mass 2m and velocity of zero) of the combination of two carts after collision?

Does energy appear to either be conserved, or not conserved, in agreement with your prediction? Do you need to worry about a potential energy associated with the velcro force? Check with an instructor at this point.

Does kinetic energy depend on the direction of motion? In other words, is the kinetic energy of the left cart before the collision the same or different from the kinetic energy of the right cart?

Since kinetic energy does not depend on direction, it remains just a number (with units) in two and three dimensions. This is true in general of all forms of energy. It is what we call a scalar quantity; it is not broken down into components, it does not have a directional character to it.

It is critical in solving momentum and energy problems in multiple dimensions to remember that momentum is a vector, and energy is a scalar. This means that conservation of momentum actually implies *three* equations, one for each dimension, whereas energy conservation only produces one.

Since momentum is equal to mass times velocity, and velocity is already a vector, it is reasonably straightforward to produce a vector quantity for momentum. That is,

$$p_x = mv_x$$

$$p_y = mv_y$$

and

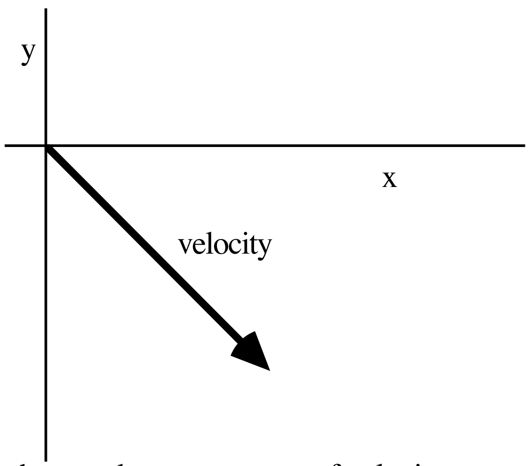
$$p_z = mv_z.$$

It is somewhat less obvious how to construct the kinetic energy in several dimensions. You will tackle this problem in two dimensions in the following exercise.

Guidebook Entry 5.10: Kinetic Energy in Two Dimensions

We will start with the assumption that the value of the kinetic energy does not depend on the direction of travel. Let's imagine that an object of 1 kg mass is traveling at 4 m/sec in the positive x direction. What is the kinetic energy of this object?

Imagine now that this object is moving at the same speed in a direction that is 45° "below" the x axis, as shown:



What are the x and y components of velocity, v_x and v_y?

Can you use Pythagoras to relate the speed to v_x and v_y?

Given this relationship, what is the expression for the kinetic energy in terms of the components of velocity?

If the velocity is in the opposite direction, the sign of each of the components reverses. Show that the kinetic energy, however, remains the same.

This expression is easily generalized to three dimensions to the expression

$$K = \frac{1}{2}m\left(v_x^2 + v_y^2 + v_z^2\right).$$

Because Pythagoras tells us that the magnitude of the velocity (the hypotenuse, if you like) is given by

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2},$$

this is usually just written like the one dimensional version,

$$K = \frac{1}{2}mv^2.$$

To gain a little practice with components and the kinetic energy expression, we will take the familiar example of projectile motion. You will wish to recall that the components of velocity and position for an object in projectile motion are those appropriate for zero force in the horizontal direction

$$v_x = v_{xo} \text{ and } x = x_o + v_{xo}t$$

and those appropriate for constant acceleration -g for the y direction

$$v_y = v_{yo} - gt \text{ and } y = y_o + v_{yo}t - \frac{1}{2}gt^2.$$

Guidebook Entry 5.11: Kinetic Energy in Projectile Motion

For objects that are subject only to the gravity force, do you expect energy to be conserved? Explain.

An object of mass 100g is thrown with an initial x velocity of 2 m/sec and a y velocity of 10 m/sec. How long does it take for the object to reach its maximum height (that is, when its vertical velocity goes to zero)?

How high will the object have traveled in that time?

What is the kinetic energy at the beginning?

What is the kinetic energy at the top?

What is the potential energy at the top?

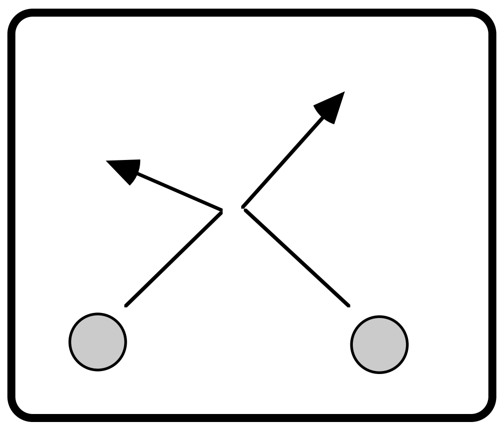
Is the total energy conserved in this case? This can be a tricky calculation, so make sure to check your answer with an instructor.

Session 4: Collisions on Air Tables—Practice with Energy and Momentum

We often use examples like hockey pucks moving frictionlessly on ice. While we don't have a convenient hockey rink to test these ideas out experimentally, we do have some equipment that is a pretty decent approximation. In this lab, you will get some experience with calculating kinetic energy and momentum from the motion of metal pucks that float on a cushion of air over a piece of paper. These pucks are connected to a high voltage "spark generator" that allows them to make regular marks on a large sheet of paper to record their motion.

Guidebook Entry 5.12: Collisions of Two Pucks: Predictions

Go and look at the metal pucks that we are about to bump into one another. Clunk them into one another so that you have some idea of what the force feels like. This force will be the only significant force acting; the pucks will float almost frictionlessly from the cushion of air. We are going to look at collisions that look qualitatively like the sketch below:



Do you expect **momentum** to be conserved in this situation? Explain.

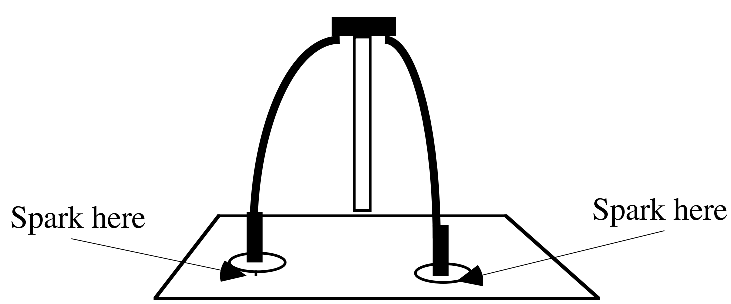
Do you expect **kinetic energy** to be conserved in these kind of collisions? Explain.

We also have some velcro rings to attach to the pucks, which makes them stick after colliding. Look at these as well.

Do you expect that the velcro pucks will conserve **momentum** better, worse, or the same as the original metal pucks? Explain.

Do you expect that the velcro pucks will conserve **kinetic energy** better, worse, or the same as the original metal pucks? Explain. Then, make sure to discuss all of the above predictions with an instructor.

The air tables work in a rather clever way. Rather than have air pushed up from the table, the pucks have an air supply that comes from the top through two surgical rubber tubes. In addition, there is a conducting chain running through the tubes that brings the electrical signal down to a metal point in the center of the bottom of each puck. Under the paper is a carbon impregnated mat that can conduct the electric spark. The high voltage spark is applied from one chain to the other, which causes the electric current to run down one chain, through the paper (where it splashes a little bit of the carbon on the underside of the paper), through the carbon mat to the other point on the other puck, and then back up the second chain. Because of this, it is essential that the pucks stay on the table whenever the spark generator is turned on, otherwise it is possible to shock yourself. While this shock is not harmful, it is quite unpleasant. The spark generator should be set on 20 sparks per second. After the basic unit is turned on, the sparks are initiated by stepping on the pedal, during which you will hear a soft clicking sound. If you hear a loud snapping sound, release the pedal because this is indicative of a large spark gap, and a risk of a shock.



Guidebook Entry 5.13: Collisions of Two Metal Pucks:

Place a fresh sheet of paper on one of the air tables with two metal pucks. Turn on the air supply, and practice a few collisions before turning on the sparker. When you are able to make a nice collision that is clearly two dimensional (and not linear), then turn on the sparker, and record a collision. Turn the sparker off just after the first puck hits a side. Before you turn the paper over to look at the dots, take a pen and mark the initial motion so that you will be able to know which end is the beginning. Once having done so, you can remove the paper to analyze it.

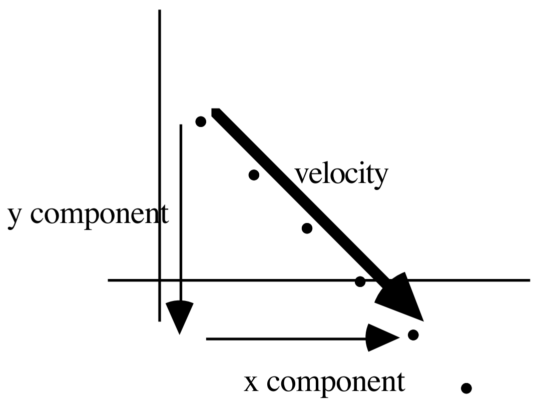
Since the dots are placed every 20th of a second, you can easily determine the velocity with a ruler. It is more accurate to calculate velocity from a span of several dots than just between two adjacent dots.

What is the initial kinetic energy of each puck?

What is the final kinetic energy of each puck?

Was kinetic energy conserved? How close to energy conservation did you come? Was this what you predicted?

Now define a coordinate system on the sheet of paper. It is much better if you choose one of the initial velocities to define the x direction. The y direction must then be perpendicular to the x direction. Draw and label these axes clearly on your paper. You may then use triangles to find the components of velocity in each component.



MOMENTUM BEFORE:
Record the momentum of the first puck before:

x momentum: y momentum:

Record the momentum of the second puck before:

x momentum: y momentum:

Record the total momentum before:

total x momentum: total y momentum:

MOMENTUM AFTER:
Record the momentum of the first puck after:

x momentum: y momentum:

Record the momentum of the second puck after:

x momentum: y momentum:

Record the total momentum after:

total x momentum: total y momentum:

Is momentum conserved? Do you observe what you predicted?

Why did you have to define a coordinate system to check momentum conservation, but didn't have to do so to check energy conservation?

You should have observed that momentum was reasonably conserved (since there are no external forces) and that there was a small but noticeable energy loss (which we might have suspected, since the force of collision might have involved some non-spring-like components that lead to denting of the pucks or other friction-like forces). The velcro pucks should have some significantly different behavior.

Guidebook Entry 5.14: Collisions of Two Velcro Pucks:

Repeat your collision measurements with the velcro pucks.

How well is kinetic energy conserved? Write the relevant data below.

Is momentum still reasonably conserved? Write your data below.