Unit 3: Newton's Second Law

In the last unit, you found the important role that acceleration plays in the motion of objects. In particular, you discovered that acceleration is directly proportional to the amount of force applied to the object, with a proportionality constant of 1/m, or in terms of an equation, F=ma. We have some intuitive sense of what this force concept means if we are actually pushing on an object, but it becomes a bit more difficult if the force is exerted by some other object, such as a spring, or gravity. In practice, when we are making theoretical predictions of motion, we need more than just F=ma; we also need some rule that tells us what the force is. By then connecting the force law to F=ma, we have an equation that tells us how the motion is changing (through the acceleration a) in relation to what it is already doing (generally in terms of x and v). We have seen such a force law already: we found for the spring that the force was proportional to the stretch of the spring. We will expand on this by developing several different force laws for various situations, and then by connecting the force law with Newton's second law F=ma, produce a differential equation (i.e. an equation with a derivative expression in it) that describes the motion.

Session 1: Force Laws

The simplest possible force laws give the force in terms of some parameter of the motion. Probably the four simplest are:

Force is zero
Force is a non-zero constant
Force is proportional to position
Force is proportional to velocity.

In this session we will consider forces of each type. Some of these you have seen already, and for others we will look at new systems. For each of them, you will also consider what the qualitative behavior of the motion is for that particular force law.

Guidebook Entry 3.1: No Force

Give an example of a system in which there is no [net] force.

If we believe F=ma, what must the acceleration be if the force is zero?

Recall that the acceleration is the time derivative of the velocity–it is the time rate of change of the velocity function. What then is the rate of change of the velocity if the force is zero?

Describe in your own words the motion of an object with zero force applied to it.

The candlepin bowling balls rolling in the hallway make a good approximation of zero force acting on an object. How could you verify this? Describe an experiment that might test this.

Take a bowling ball and whatever equipment you need and attempt to verify your predictions for the zero force situation. How well does the candlepin ball fit the zero force model?

Under Newton's new system, motion with zero force was fundamentally different from the Aristotelian system that had dominated scientific thought for centuries. It was so different that the notion of constant velocity under zero force is known as Newton's first law, even though it is also encompassed in his second law as you saw above.

The second category of simple force laws is that of a constant force.

Guidebook Entry 3.2: The Constant Force

We have already seen one case in which the force on an object is non-zero, but does not depend on the position or velocity of the object: the gravity force (at least here near the surface of the earth).

Use the force gauge (spring scale), or the force probe to verify that this is true. Does a hanging weight pull any harder if:

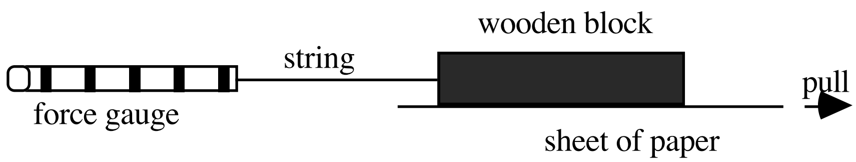
it is up high than if it is down low?

if it is moving upward at a constant rate than if it is still (you may want to try this in the elevator)?

What do you see on the spring scale if you accelerate the weight? Is this indicative of a *gravity* force change? Why or why not? (You may need to discuss this with an instructor).

Describe motion under a constant force—you may use freefall as a case to consider. If you saw a distance versus time graph, how would you recognize this as resulting from a constant force?

The gravity force near the surface of the earth is extremely constant. Another force that is approximately constant is the force of sliding friction. To investigate the behavior of this force, use the following equipment:



There is frictional force between the block and the underlying sheet of paper. If we try to pull the paper out from underneath the wooden block, we read a force on the spring gauge. This force is just large enough to keep the block from moving, which means it matches the frictional force.

Pull the paper out slowly while one of your partners holds the force gauge and monitors the reading (or you may use the force probe and Logger Pro). What is the force?

Now pull the paper out more quickly. Does the force vary appreciably with the velocity?

Add some weight on top of the block. Does the frictional force vary with the weight?

Try holding the paper still, and pulling the block over the paper. Is the force the same or different than it was when you pulled the paper out from underneath?

The usual approximate law for sliding friction gives the frictional force as

$$F_{friction} = \mu N_{contact}$$

where F is the friction force, N is the force of contact between the two sliding surfaces (in our case above, this is just equal to the weight of the block), and μ is a constant known as the coefficient of friction.

Guidebook Entry 3.3: Details of the Friction Force Law

To what extent is the friction force law an accurate representation in terms of the dependence of the frictional force on the contact force? Is the frictional force linearly dependent on the weight? Take several measurements, and graph the frictional force versus weight. (Make sure to make this a graph versus *weight*, not mass. Weight is the force, that is, mg, and not the mass.) What do you conclude?

What is an approximate value of the coefficient of friction for your block on the paper? What are the units of the coefficient of friction?

We have already investigated a force that is directly proportional to distance: the spring force law. If we define our origin to be the position at which the force should be zero, this is given simply by Hooke's Law:

$$F=-kx$$

where k is a constant known as the spring constant.

Guidebook Entry 3.4: Forces Proportional to Position

Describe qualitatively the sort of motion that results from a force law of the form $F = -kx$.

Would you expect the motion to be any different if the force law were given by $F = +kx$? Discuss this with your partners and an instructor.

Finally, we want to consider a force law where the force is proportional to the velocity. This sort of force occurs for objects that are moving through a thick fluid. We will make measurements of this using the force probe, and a small weight which we drag through some diluted corn syrup.

Guidebook Entry 3.5: A Drag Force

Because it is difficult to move the force probe smoothly, we will again use the trick that we used with sliding friction, and we will keep our object of interest still, and move the medium that is causing the friction, or in this case fluid drag. Hang a 50 g weight from the calibrated force probe with a thin string. Fill a plastic cylinder with diluted corn syrup. Lift the cylinder up so that the weight drops into it. Zero the force probe. Now, as you move the cylinder up and down, you should be able to observe changes in the force readings on Logger Pro. Roughly how big are they?

Now, while one of you keeps the cylinder in place, position your motion detector under the cylinder. **BE CAREFUL NOT TO DRIP CORN SYRUP INTO THE DETECTOR!!** Start Logger Pro, and move the cylinder up and down. Make sure to move slowly enough so that the string always remains taut. Analyze the resulting force. Does it appear to be proportional to position, velocity, or acceleration? Describe how you tested for this.

If the force law is described by the equation

$$F=\square v$$

where □ is a constant, should □ be positive or negative? Explain your answer.

Session 2: Motion for Constant Forces

In this unit, we will investigate several different types of motion that arise from constant, or near constant, forces. As you will discover, all of these forces give rise to very similar motions. This is exactly what is predicted by Newton's second law, which for a constant force, predicts constant acceleration.

First, we will use Newton's second law to predict the general form of motion under a constant force, then we will look at a variety of different motions, and see if this prediction is validated by actual motion. We saw for the case of zero force that this resulted in a constant velocity. Zero force (and hence zero acceleration) did not determine what that velocity was, however. The actual value of the velocity was determined by our starting conditions. Zero acceleration also did not determine the actual value of the position; we could choose to start at any location. However, once we determined the initial position and velocity, then the full motion was determined. In terms of equations, this would be expressed as:

$$v(t) = v_o$$
$$x(t) = v_o\, t + x_o \quad (\text{zero accel})$$

where x_o is the initial position and v_o is the initial velocity. We will find the same is true for other force laws--we will write a general form to the solution, and then a particular solution is determined by the initial position, velocity, and acceleration.

Guidebook Entry 3.6: Theoretical Predictions

Each derivative of a polynomial decreases the exponent of each term by one. This suggests that the algebraic form of the position as a function of time should be

$$x(t) = bt^2 + ct + d$$

where b, c, and d are some constants. Use your knowledge of calculus to show that this does produce constant acceleration.

If the value of the acceleration is "a," can you express b, c, or d in terms of a?

What is the initial position of the object, that is $x(0)$, which we also denoted earlier as x_o in terms of b,c, and/or d?

What is the velocity as a function of time? [You may already have calculated this as an intermediate result when you were looking at acceleration.]

What is the initial velocity, v_o in terms of b,c, and/or d?

Rewrite the equations for position as a function of time and velocity as a function of time using a, v_o and x_o instead of b, c, and d. Check with an instructor to verify your result.

These general results are extremely useful, as they apply to all constant force situations. In the next exercises, we will apply these equations to several motion cases.

Our first example is not really a constant force at all, but one in which we ignore the rapid variations in force. To be specific, we will take the candlepin balls, and hit them repeatedly with a long rod. To the extent that each hit is the same, and if we look at behavior on a time scale greater than the time between hits, we will see behavior that looks like constant force, and therefore constant acceleration.

Guidebook Entry 3.7: Bonking Bowling Balls

In this exercise, technique is important for good results. There are two important tasks. The first involves the marking of the position of the ball at regular time intervals. This is accomplished by dropping bean bags next to the ball once per second. The second technique is the ball-bonking. This should be done frequently (about 3 times per second) and gently (you should only be swinging the tip of the stick maybe 10 or 20 cm). The hallways give the best smooth surface for this, as well as sufficient room. Practice these techniques a few times before trying to take actual measurements.

Once you have found who in your group is best at each task, you should make measurements of motion in which the initial velocity is zero. It is easiest to simply count or clap at a regular rhythm rather than trying to read the watch--although you can use the watch at first to "calibrate" your rhythm. Start with the ball at rest. Start clapping or counting, and then simultaneously start bonking the ball and dropping bean bags. This will give you an array of bags on the floor. Measure their positions relative to the starting point using one of the two-meter sticks. Record your results in the table below:

time	position

Now, repeat this measurement with an initial velocity opposite to the direction you are hitting the ball. This will take some practice so that you can keep up with the ball, and also keep the earlier bags separate from the later bags. Keep hitting the ball even if it momentarily comes to rest. Practice this a couple of times, and then record your results below:

time	position

Use Excel to graph each of these sets of data as position as a function of time. Do they both have (qualitatively at least) the expected quadratic behavior that your theoretical analysis would predict? Attach your graphs, or sketch them below.

What do you predict the velocity as a function of time should look like for each of these sets of data? Sketch your predictions below.

Use Excel to calculate the average velocity in between bean-bag drops (much as you did for the video analysis of free-fall). Graph your data and attach a copy. Does it look like your prediction?

Use the slope of the velocity versus time graphs to determine the acceleration. Were the two accelerations close to one another?

The bowling ball exercise is nice in that you have a direct feeling for applying the force to the object. The obvious disadvantage is that you are not so effective at keeping this force constant. However, gravity is very good at producing a constant force. In the next exercise, we will use gravity and low-friction carts to produce a constant acceleration.

Guidebook Entry 3.8: Rolling Carts on an Incline

First, we will use the cart in a force-free situation. What should the position versus time graph look like?

Use Logger Pro to measure this case. Place the motion detector to the end of your track (the laminated board). Take data as you roll the cart slowly away from the motion detector, and note that there is a minimum distance which the motion detector will register. Give the cart a brief push at the beginning, and then let it roll freely. Does your result agree with your prediction?

If this is genuinely a zero force situation, the velocity should be constant. Look at the velocity versus time graph. Can you explain any discrepancies?

Now take your track and prop up one end a few cm. What direction is the net force on the cart?

Use the force probe to measure the force on the cart, and record this below. Use a short piece of string to attach the cart to the probe. The force will be small, so calibrate with a small force, say 1 N or so.

Measure the mass of the cart on a balance. What do you predict the resulting acceleration should be?

Release the cart from rest at the top, and use Logger Pro to record the position versus time. Graph the position and velocity versus time. Sketch them below, or attach them. Are they consistent with constant acceleration?

What is the measured acceleration of the cart? (You can do this all from within Logger Pro—you may wish to print your graph of velocity versus time, or use the fitting routine in the analysis menu.) Explain how you found it. Is this close to your prediction?

Predict what position and velocity graphs should look like if you start the cart moving up the track:

Now repeat your measurements with an initial velocity up the ramp. Do they agree with your prediction?

What should the acceleration be at the peak of the motion (minimum distance value)? Explain.

Look on the acceleration versus time graph. Is the acceleration at the peak what you predicted?

What do you expect to happen to the motion, and in particular the acceleration, if you increase the mass of the cart?

You can increase the mass by adding one of the metal bars, which exactly doubles the mass of the cart. Do this, and then measure the acceleration. How does this compare to the previous acceleration? Is it what you predicted?

We can also predict what the force down the plane should be, if we know what the total gravitational force on the cart is. First, find the total gravity force on the cart as follows: use the force (spring) scale to determine this total force by simply hanging the cart. Does this agree with what we would predict for the gravitational force, that is F=mg, where g is the acceleration due to gravity of 9.8 m/s^2?

The force down the track should be the *component,* or projection of the force along the plane. This is given by $F_{gravity}$ sin (θ), where θ is the angle of the board relative to the table (this angle is hard to measure with a protractor, but the result is easy using similar triangles--you may wish to ask for help). Does this agree with what you measured?

As a final example, we will consider motion subjected to the force of sliding friction.

Guidebook Entry 3.9: Sliding Friction

Place a friction pad on your cart, and put your board back down to level.

Use the force probe to determine the force of sliding friction as you roll the cart over the track, and record this below. (You may need to re-calibrate with a small force, say 1 N, first.)

What acceleration do you expect to see when the cart is sliding to a stop?

Roll the cart slowly so that it comes to a stop on the track. Use Logger Pro to record this motion. Does the acceleration appear to be nearly constant? Does the value agree with your prediction?

Session 3: Numerical Solutions and Changing Forces

In this session we will do a fair amount of theoretical analysis of the solutions to force laws that are not constant. We'll do some of this with calculus wherever we can find exact solutions. However since these are exact solutions to force laws that are usually approximate (e.g. the drag force law and spring force law), the term "exact" must be taken with a grain of salt. We will also investigate numerical (that is, computer generated) solutions to these equations using Excel. Numerical solutions to physics problems have become so important that the traditional division of physicists into theorists and experimentalists no longer suffices, and a new term has emerged: the computationalist. While the techniques that we will use here are very simple, they are still of general utility, and share a great many features with the fancier "algorithms" that computationalists use. In addition, since you will use Excel to produce these results, you don't need to learn a programming language, and you can use them on any computer system that has a spreadsheet program (which includes virtually every Mac and PC on campus). You may find these computational techniques useful, since simple rate (or differential) equations pop up not only in math and physics, but in many disciplines such as biology, chemistry, and economics.

Guidebook Entry 3.10: Some Simple Numerical Analysis--Constant Velocity and Acceleration

You have already used Excel to produce a column of velocity values from a column of position values. Now we will use the same method in reverse to produce a column of position values from velocity values.

Here are some typical entries in an Excel spreadsheet that calculates velocities:

Table 1

	A	B	C
1	time	position	velocity
2	0	0	=(B3-B2)/0.1
3	0.1	0.5	=(B4-B3)/0.1
4	0.2	0.9	=(B5-B4)/0.1
5	0.3	1.4	

Now use your algebra skills to change this around to give us position values if we know what the velocity values are. In particular, what formulae would go in cells B3, B4, and B5 to give the correct positions?

Table 2

	A	B	C
1	time	position	velocity
2	0	0	5
3	0.1		4.9
4	0.2		5.1
5	0.3		5

Make sure that the equations make references to the cells in the velocity column (e.g. C2), rather than specific values.

If you are having trouble with this, consider cell B3, which is the first empty one. In Table 1, note that cell C2 has a formula in it that gives C2=(B3-B2)/0.1 (where 0.1 is the time step between cell values). Rearrange this for B3, and write the resulting equation in cell B3. Do likewise for the succeeding cells.

Now it is time to create an actual Excel spreadsheet with columns giving time, position, and velocity. Let the time run from 0-10 seconds, and let the velocity be constant.

What should a graph of position versus time look like if velocity is constant? Sketch it below:

Now insert the same value for the velocity (say 5 m/s, although don't forget that you cannot enter units in the cell and still be able to do arithmetic) in each cell in the velocity column. Using your formula developed above, fill in the position column. Graph the resulting position values against time. Sketch your result below, or attach a copy. Do you get what you expected?

Now let's make our example a bit more complicated. Imagine that we had constant acceleration. If the velocity starts out at zero, and the position is initially zero, what should the position versus time graph look like for positive acceleration?

In this case, the velocity might start out at zero, and increase 0.2 m/s for each time interval of 0.1 s. To reflect this, make the first entry in the velocity column 0, the next one 0.2, the next 0.4, and so on. Graph the position versus time. Does it have the right qualitative behavior? Discuss any discrepancies with an instructor.

What is the acceleration in this instance if the velocity changes 0.2m/s in 0.1 s?

Given the constant acceleration formula you discovered in the last activity guide:

$$x(t) = \frac{1}{2}at^2 + v_o t + x_o$$

what should the position be after 10 seconds of motion, if the initial position and velocity are both zero?

How closely does this agree with the spreadsheet calculation? Discuss any discrepancies with an instructor.

In this last calculation, we simply inserted velocity values by hand. However, we can be much slicker than this. We know that acceleration bears the same relationship to velocity that velocity bears to position. So you can use the same formula to update a velocity that you used to update a position, if you simply replace the analogous quantities. Create an acceleration column, and insert the constant value 2 into each cell in this column. Then insert the appropriate formula into each cell in the velocity column to calculate each velocity value from the previous velocity, acceleration, and time interval of 0.1 s.

If this seems a bit obscure, remember that you can calculate an average acceleration as the change in velocity divided by the change in time. In the context of this example, it would be

$$a = \frac{v_2 - v_1}{0.1\,\mathrm{sec}}.$$

Rearrange this to find v_2 in terms of the other quantities. This is the formula that you need to put in each cell of the velocity column. Make sure that the reference to "a" is in terms of the current cell entry in the acceleration column, not just to the number 2. If you are at all unsure, check with an instructor before proceeding.

Graph your resulting position versus time; is it the same as before?

Keep this spreadsheet that you have developed. You will be able to use it to calculate the motion of an object subjected to almost any force. To have a concrete example, we will look at the drag force, $F = -\square v$.

Guidebook Entry 3.11: Motion Under the Drag Force

Imagine that an object starts out with a large velocity, but is acted upon by a resistive (force is opposite to the direction of motion) drag force. A real (albeit approximate) example is that of a water skier coming to rest after she has let go of the tow rope. With this in mind, sketch qualitatively what you expect the graphs of position versus time graph and velocity versus time would look like:

We can observe this experimentally with Logger Pro. We need something very light that moves with minimal friction, but suffers considerable drag force. Rolling Styrofoam cups (a nice invention of Professor Case) work quite well for this purpose. On a level track, roll the mated cups in front of the motion detector. Experiment a bit with initial velocities to get nice position versus time and velocity versus time graphs. Do they look like your predictions for the water skier?

Now let's give a little extra thought to your graphs. From your velocity versus time graph, draw a qualitative acceleration versus time graph:

Remember that acceleration is proportional to force, and the drag force in particular is large when the velocity is large. Do your sketches agree with this? Which graphs are qualitatively different from the constant acceleration graphs?

Now you can insert this drag force law into your Excel spreadsheet, and see how well your numerical solution agrees with your intuitive guesses. Let's use the simplest possible values. If the force is proportional to the velocity, and the acceleration is proportional to the force, let's choose constants such that

$a = -v$. (i.e. □/m = 1)

Use this relationship to fill in the formulas in the acceleration column. Choose your initial position to be $x = 0$, and the initial velocity to be $v = 10$. Again, use a time interval of 0.1 s. Graph position versus time and velocity versus time. Sketch your results below, or attach your graph. Do they agree with your guesses?

We can also get an analytical solution in this case. If we combine Newton's second law

$$F = ma$$

with the drag force law

$$F = -\square v$$

we get the relationship

$$ma = m\frac{dv}{dt} = \square\square v, \text{ or}$$

$$\frac{dv}{dt} = \square\frac{\square}{m}v.$$

We know that the exponential is a function that returns itself upon differentiation. Using a trial velocity function

$$v = \square e^{\square t},$$

show that this satisfies the above equation, by direct substitution. Does this solution place any requirements on $\square$ or $\square$?

Given an initial velocity of v_0 does this place any requirements on $\square$ and $\square$?

Does this solution qualitatively agree with your numerical solution?

Although we won't deal with it now, the general form of a solution for position versus time in the presence of a drag force is

$$x(t) = x_{final} \square \frac{mv_o}{\square} e^{\left(\square\frac{\square}{m}t\right)}.$$

The constant x_{final} is the final resting point of the object (x at t = infinity, in contrast to our usual practice of solving for constants in terms of initial position--it is just a little neater expression.) It would be a good calculus exercise to show that this gives the correct velocity function, and does also agree with the original differential equation.

In the next exercise you will use your spreadsheet to look at the mass on a spring.

Guidebook Entry 3.12: The Spring Force

You have seen the behavior of an object subject to a spring force law $F = -kx$. Sketch what the position versus time graph should look like:

We know that the force for a spring is proportional to position (F=-kx, which is our force law), and the acceleration is proportional to the force (F=ma, Newton's second law). Assume that the mass and k are such that these two relationships combine to give

$$a = -\frac{x}{2}.$$

Place the appropriate set of equations in your acceleration column, and see if you get the correct qualitative behavior of the position versus time solution. Attach a graph, or sketch it below:

As we will see later, the differential equation in the spring case requires a solution whose second derivative is the same function back again, only with a negative constant out front. The functions with this property are sine and cosine, which agree with the observed and calculated oscillatory behavior.

I carefully chose the constants for this example so that you would see only about one period of the oscillation. It turns out that this particular numerical method is a bit weak for the spring force, it gives a solution whose amplitude increases with time! There are better numerical methods to use for this case that are only slightly more complicated to employ. Nevertheless, even our very simple method produces the correct qualitative behavior, and gives a good value for the period of the oscillation.

To show the general utility of our numerical solution method, let's apply it to a new force law, one that combines a drag force with a constant force. This describes quite well a very light falling object, such as a crumpled piece of paper.

Guidebook Entry 3.13: Falling Can Be a Drag!

Let's first look at this example numerically with Excel. Replace the acceleration column entries in your spreadsheet with an acceleration that might result from a constant force (e.g. gravity) plus a drag force (air drag on the paper). To be concrete, let

$$a = 1 - v.$$

Be sure to put v=0 as your initial velocity. Look at position graphed against time, and velocity graphed against time. Sketch them below. Where does the motion look like freefall? Where does it look like zero force motion?

Now let's try to observe this motion experimentally. Crumple up a sheet of paper to a diameter of roughly 10 cm. Put the motion detector on the floor, and drop the paper wad from near the ceiling (you may stand on a table to do this). It make take some practice to get some good data, and you may need to recrumple the paper. Do you see qualitative similarities between your numerical result and your experimental numbers? Explain.

Another good example of this is rolling the double Styrofoam cups down the inclined plane. Try this at two different angles; compare the results.

The terminal velocity is the final, constant velocity reached when the gravitational force and the drag force are equal. If the drag force really is proportional to velocity, the terminal velocity should be proportional to the gravitational force. For our inclined plane example, this should be proportional to the sine of the angle of incline of the board. Does this appear to be the case?

Look at other force laws?--Friction, spring, gravity, viscous drag
Prediction from F=ma
No Force & Constant Force--Ball bonking

Session 2: No Force & Constant Force--Rolling Carts, Inclined Planes, Why does everything fall at the same rate? (I.e. change mass on cart, no effect)
Friction pads
Mass effect with friction force

Session 3: Spring Force--Prediction from F=ma
Observation of Mass-Spring system